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Chiral compactification on a square

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Abstract: We study quantum field theory in six dimensions with two of them compactified on a square. A simple boundary condition is the identification of two pairs of adjacent sides of the square such that the values of a field at two identified points differ by an arbitrary phase. This allows a chiral fermion content for the four-dimensional theory obtained after integrating over the square. We find that nontrivial solutions for the field equations exist only when the phase is a multiple of $\pi/2$, so that this compactification turns out to be equivalent to a T^2/Z_4 orbifold associated with toroidal boundary conditions that are either periodic or anti-periodic. The equality of the lagrangian densities at the identified points in conjunction with six-dimensional Lorentz invariance leads to an exact $Z_8 \times Z_2$ symmetry, where the Z_2 parity ensures the stability of the lightest Kaluza-Klein particle.

Keywords: Field Theories in Higher Dimensions, Space-Time Symmetries.

Contents

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1. Introduction

Quantum field theory in six dimensions has been studied in connection to various alternatives for physics beyond the standard model, and was shown to provide explanations for proton stability [1], the origin of electroweak symmetry breaking [2]–[5], the number of fermion generations [6]–[11], and the breaking of grand unified gauge groups [12, 13, 14].

An important question is how are the two extra dimensions compactified. If any of the standard model fermions propagate in the extra dimensions, a restriction on the compactification comes from the requirement of having chiral fermions in the four-dimensional low-energy theory. Only few examples of compactifications with this property have been analyzed in detail so far. The T^2/Z_2 orbifold is a a parallelogram folded once onto itself (see, e.g., [15]). The field decomposition in Kaluza-Klein (KK) modes is given in [16] for the case where the parallelogram is a rectangle. The T^2/Z_4 orbifold, which is a compactification on a square, has the merit of automatically preserving a Z_8 subgroup of the six-dimensional Lorentz symmetry [1] which ensures a long proton lifetime, and forces the neutrino masses to be of the Dirac type [17]. In a different context, the T^2/Z_4 orbifold was shown to allow the Higgs doublet be part of a G_2 gauge field [4]. A discussion of T^2/Z_N orbifolds in general can be found in [5] while more general manifolds with conical singularities were considered in [18].

In the case of five-dimensional theories, any orbifold compactification is equivalent to a set of chiral boundary conditions (see, e.g., [19]), while the reverse is not true [20, 21]. For six-dimensional theories, however, neither the boundary conditions associated with various orbifolds nor the general restrictions imposed by the action principle on the boundary conditions have been presented in the literature.

In this paper we explore a chiral compactification on a square. We derive a simple set of boundary conditions, and prove that it allows nontrivial solutions to the field equations only when it is equivalent to the T^2/Z_4 orbifold. We also determine the KK wave functions for scalars and fermions, and identify the unbroken subgroup of the six-dimensional Lorentz symmetry.

The boundary conditions and the solutions to the field equations for free scalar and fermion fields are analyzed in sections 2 and 3. Section 4 is devoted to the symmetries of the action. Interactions among KK modes are studied in section 5, while the connection to orbifolds is presented in section 6. A summary of results is given in section 7.

2. Compactification on a square

The spacetime considered in this paper is six-dimensional: four spacetime dimensions of coordinates x^μ, $\mu = 0, 1, 2, 3$, form the usual Minkowski spacetime, and two transverse spatial dimensions of coordinates x^4 and x^5 are flat and compact. We analyze a simple compactification: a square with $0 \le x^4, x^5 \le L$.

In this section we study the behavior of free scalar fields, $\Phi(x^\mu, x^4, x^5)$, described by the following action:

$$S_\Phi = \int d^4x \int_0^L dx^4 \int_0^L dx^5 \left(\partial_\alpha \Phi^\dagger \partial^\alpha \Phi - M_0^2 \Phi^\dagger \Phi\right) . \tag{2.1}$$

We use letters from the beginning of the Greek alphabet to label the six-dimensional coordinates $\alpha, \beta, \ldots = 0, 1, \ldots, 5$, and letters from the middle of the Greek alphabet to label the Minkowski coordinates $\mu, \nu, \ldots = 0, 1, 2, 3$.

Under a variation of the field, $\delta\Phi(x^\mu, x^4, x^5)$, the variation of the action is given by

$$\delta S_\Phi = \delta S_\Phi^v + \delta S_\Phi^s , \tag{2.2}$$

where the first term is a "volume" integral,

$$\delta S_\Phi^v = -\int d^4x \int_0^L dx^4 \int_0^L dx^5 \left(\partial^\alpha \partial_\alpha \Phi^\dagger - M_0^2 \Phi^\dagger\right) \delta\Phi , \tag{2.3}$$

and the second term is a "surface" integral,

$$\begin{aligned}\delta S_\Phi^s = \int d^4x \Bigg[&\int_0^L dx^4 \left(\partial_5 \Phi^\dagger \delta\Phi \Big|_{x^5=L} - \partial_5 \Phi^\dagger \delta\Phi \Big|_{x^5=0} \right) + \\ &+ \int_0^L dx^5 \left(\partial_4 \Phi^\dagger \delta\Phi \Big|_{x^4=L} - \partial_4 \Phi^\dagger \delta\Phi \Big|_{x^4=0} \right) \Bigg] .\end{aligned} \tag{2.4}$$

Here we have assumed as usual that the field vanishes at $x^\mu \to \pm\infty$.

Given that the action has to be stationary with respect to *any* variation of the field, the volume and surface terms must vanish independently. Requiring $\delta S^v_\Phi = 0$ implies that Φ is a solution to the six-dimensional Klein-Gordon equation, while $\delta S^s_\Phi = 0$ forces the boundary conditions that can be imposed on Φ to obey a certain restriction. To derive it, we rewrite the surface integral as

$$\delta S^s_\Phi = \int d^4x \int_0^L dy \left(\frac{\partial \Phi^\dagger}{\partial y}\delta\Phi\bigg|_{(x^4,x^5)=(L,y)} + \frac{\partial \Phi^\dagger}{\partial y}\delta\Phi\bigg|_{(x^4,x^5)=(y,L)} - \right. \\ \left. - \frac{\partial \Phi^\dagger}{\partial y}\delta\Phi\bigg|_{(x^4,x^5)=(0,y)} - \frac{\partial \Phi^\dagger}{\partial y}\delta\Phi\bigg|_{(x^4,x^5)=(y,0)} \right) = 0\,. \tag{2.5}$$

The requirement that the action be stationary for *arbitrary* variations of the field satisfying certain boundary conditions implies that the integrand vanishes point by point:

$$\partial_4\Phi^\dagger\delta\Phi\Big|_{(x^4,x^5)=(L,y)} + \partial_5\Phi^\dagger\delta\Phi\Big|_{(x^4,x^5)=(y,L)} = \partial_4\Phi^\dagger\delta\Phi\Big|_{(x^4,x^5)=(0,y)} + \partial_5\Phi^\dagger\delta\Phi\Big|_{(x^4,x^5)=(y,0)} \tag{2.6}$$

for any $y \in [0, L]$.

Even with this restriction, there are many possible boundary conditions. Identifying the opposite sides of the square, which produces a torus, is an obvious example. However, it is well known that the toroidal compactification does not allow chiral fermions in the four-dimensional effective theory. A particular class of boundary conditions which allows four-dimensional chiral fermions (as shown later, in section 3) is presented next.

Consider the identification of two pairs of adjacent sides of the square:

$$(y, 0) \equiv (0, y)\,, \qquad (y, L) \equiv (L, y)\,, \qquad \forall y \in [0, L]\,. \tag{2.7}$$

Topologically, this is equivalent to folding the square along a diagonal and gluing the boundary. Note though that we choose the metric on the square to be flat. There are two points (see figure 1) that remain invariant under the folding operation (2.7): $(0, 0)$ and (L, L).

We interpret the identification of different sides of the square as the requirement that the lagrangian density at points identified by the folding operation (2.7) is the same:

$$\mathcal{L}(x^\mu, y, 0) = \mathcal{L}(x^\mu, 0, y)\,, \\ \mathcal{L}(x^\mu, y, L) = \mathcal{L}(x^\mu, L, y)\,, \tag{2.8}$$

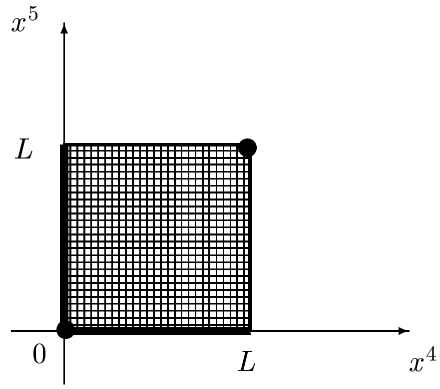

Figure 1: Square compactification with identified pairs of adjacent sides.

where $\mathcal{L}$ is the integrand shown in eq. (2.1). This ensures that the physics at identified points is the same. On the other hand, the field at two identified points does not need to be the same. The global U(1) symmetry of the lagrangian suggests that the field values at two identified points may differ by a constant phase:

$$\Phi(x^\mu, y, 0) = e^{i\theta}\Phi(x^\mu, 0, y)\,, \tag{2.9}$$

for any $y \in [0, L]$. Taking the derivative with respect to y in the above equations we find that

$$\partial_4 \Phi|_{(x^4,x^5)=(y,0)} = e^{i\theta}\ \partial_5 \Phi|_{(x^4,x^5)=(0,y)} \,. \tag{2.10}$$

Using these conditions we obtain

$$\mathcal{L}(x^\mu, 0, y) - \mathcal{L}(x^\mu, y, 0) = \partial_5 \Phi^\dagger \partial_5 \Phi\Big|_{(x^4,x^5)=(y,0)} - \partial_4 \Phi^\dagger \partial_4 \Phi\Big|_{(x^4,x^5)=(0,y)} \,. \tag{2.11}$$

Therefore, the lagrangian is the same at two identified points only if, in addition to eq. (2.9), the following condition on the derivatives is satisfied:

$$\partial_5 \Phi|_{(x^4,x^5)=(y,0)} = e^{i\theta'}\ \partial_4 \Phi|_{(x^4,x^5)=(0,y)} \,. \tag{2.12}$$

This has a simple geometrical interpretation: the folding of the square is smooth, i.e., the derivative of the field in the direction perpendicular to the identified boundaries is continuous up to a phase.

Conditions analogous to (2.9) and (2.12) have to be imposed on the other two sides of the square, but the phases may be different both for the fields,

$$\Phi(x^\mu, y, L) = e^{i\tilde{\theta}} \Phi(x^\mu, L, y) \,, \tag{2.13}$$

and for the derivatives,

$$\partial_5 \Phi|_{(x^4,x^5)=(y,L)} = e^{i\tilde{\theta}'}\ \partial_4 \Phi|_{(x^4,x^5)=(L,y)} \,. \tag{2.14}$$

Although the phases θ, $\tilde{\theta}$, θ' and $\tilde{\theta}'$ may be different in general, there are certain constraints on them. Most importantly, the boundary conditions must be consistent with the stationarity of the action, i.e., the general condition (2.6). Given that the variation $\delta\Phi$ on the boundary has to obey the same constraints as Φ, namely eqs. (2.9) and (2.13), we can use eqs. (2.12) and (2.14) to write eq. (2.6) as

$$\left[e^{i(\tilde{\theta}'-\tilde{\theta})} + 1\right] \partial_5 \Phi^\dagger \delta\Phi\Big|_{(x^4,x^5)=(y,L)} = \left[e^{i(\theta'-\theta)} + 1\right] \partial_5 \Phi^\dagger \delta\Phi\Big|_{(x^4,x^5)=(y,0)} \,. \tag{2.15}$$

This condition must be satisfied for any field variation $\delta\Phi$, so that the left- and right-handed sides of eq. (2.15) must vanish independently. Hence, two constraints can be derived:

$$e^{i\theta'} = -e^{i\theta} \,, \qquad \text{or} \qquad \delta\Phi\ \partial_5 \Phi|_{(x^4,x^5)=(y,0)} = 0 \,, \tag{2.16}$$

and

$$e^{i\tilde{\theta}'} = -e^{i\tilde{\theta}} \,, \qquad \text{or} \qquad \delta\Phi\ \partial_5 \Phi|_{(x^4,x^5)=(y,L)} = 0 \,. \tag{2.17}$$

We now solve the six-dimensional Klein-Gordon equation,

$$\left(\partial^\mu \partial_\mu - \partial_4^2 - \partial_5^2 + M_0^2\right) \Phi = 0 \,, \tag{2.18}$$

subject to the "folding boundary conditions" (2.9), (2.12), (2.13), (2.14), with the restrictions (2.16) and (2.17). Since the boundary conditions are independent of x^μ, then Φ can be decomposed in Fourier modes as follows:

$$\Phi(x^\mu, x^4, x^5) = \frac{1}{L} \sum_{j,k} \Phi^{(j,k)}(x^\mu) f^{(j,k)}(x^4, x^5) \,. \tag{2.19}$$

The four-dimensional scalar fields $\Phi^{(j,k)}$ satisfy

$$\left(\partial^\mu \partial_\mu + M_0^2 + M_{j,k}^2\right) \Phi^{(j,k)}(x^\mu) = 0\,, \tag{2.20}$$

where $M_{j,k}^2$ is a positive eigenvalue. The $f^{(j,k)}$ functions are solutions to the two-dimensional equation,

$$\left(\partial_4^2 + \partial_5^2 + M_{j,k}^2\right) f^{(j,k)}(x^4, x^5) = 0\,. \tag{2.21}$$

A general solution to the above equation is a linear combination of eight position-dependent phases,

$$\begin{aligned} f^{(j,k)} = & \; C_1^+ e^{i\left(jx^4+kx^5\right)/R} + C_1^- e^{-i\left(jx^4+kx^5\right)/R} + C_2^+ e^{i\left(jx^4-kx^5\right)/R} + C_2^- e^{-i\left(jx^4-kx^5\right)/R} + \\ & + C_3^+ e^{i\left(kx^4+jx^5\right)/R} + C_3^- e^{-i\left(kx^4+jx^5\right)/R} + C_4^+ e^{i\left(kx^4-jx^5\right)/R} + C_4^- e^{-i\left(kx^4-jx^5\right)/R}\,, \end{aligned} \tag{2.22}$$

where j and k are real numbers such that

$$M_{j,k}^2 = \frac{j^2+k^2}{R^2}\,, \tag{2.23}$$

and we defined the “compactification radius”

$$R \equiv \frac{L}{\pi}\,. \tag{2.24}$$

The eight unknown coefficients, $C_i^\pm$, $i = 1, 2, 3, 4$, that appear in the general solution, are constrained by the folding boundary conditions. Eq. (2.9), which relates the field values on the $x^4 = 0$ and $x^5 = 0$ sides of the square, is satisfied for arbitrary j and k if and only if

$$\begin{aligned} C_3^\pm + C_4^\mp &= e^{-i\theta}\left(C_1^\pm + C_2^\pm\right), \\ C_3^\pm + C_4^\pm &= e^{i\theta}\left(C_1^\pm + C_2^\mp\right). \end{aligned} \tag{2.25}$$

The boundary condition (2.12) that relates the field derivatives at $x^4 = 0$ and $x^5 = 0$ is satisfied for arbitrary j and k provided

$$\begin{aligned} C_3^\pm - C_4^\mp &= e^{-i\theta'}\left(C_1^\pm - C_2^\pm\right), \\ C_3^\pm - C_4^\pm &= e^{i\theta'}\left(C_1^\pm - C_2^\mp\right). \end{aligned} \tag{2.26}$$

For $j = \pm k$, these eight equations are replaced by only four linear combinations of them, but in the end no new solution is allowed.

The set of eight equations (2.25) and (2.26) has to be solved subject to the constraint (2.16). For

$$e^{i\theta'} = -e^{i\theta}\,, \tag{2.27}$$

eqs. (2.25) and (2.26) have nontrivial solutions only if

$$e^{4i\theta} = 1\,. \tag{2.28}$$

This is an important restriction on the phase that relates the field values on the $x^4 = 0$

and $x^5 = 0$ boundaries. Six of the unknown coefficients are determined in terms of the remaining two, chosen to be $C^+_{1,2}$:

$$\begin{aligned} C^\pm_3 &= C^+_2 e^{\mp i\theta}\,, \\ C^\pm_4 &= C^+_1 e^{\pm i\theta}\,, \\ C^-_{1,2} &= C^+_{1,2} e^{2i\theta}\,. \end{aligned} \tag{2.29}$$

If eq. (2.27) is not satisfied, then the constraint (2.16) implies that $\delta\Phi$ or $\partial_5\Phi$ vanish at $(y,0)$, so that $f^{(j,k)}$ or $\partial_5 f^{(j,k)}$ vanish at that point. Eqs. (2.25) and (2.26) then have nontrivial solutions only if $e^{2i\theta} = 1$ and $C^\pm_2 = \pm C^\pm_1$, so that the solutions in this case are subsets of the solutions allowed by eq. (2.27). Thus, the most general solution to the two-dimensional equation (2.21) subject to the boundary conditions (2.9) and (2.12) is given by

$$\begin{aligned} f^{(j,k)}(x^4,x^5) = {}& 2C^+_1\left[e^{-i\theta}\cos\left(\frac{jx^4+kx^5}{R}+\theta\right)+\cos\left(\frac{kx^4-jx^5}{R}+\theta\right)\right]+ \\ &+2C^+_2\left[e^{-i\theta}\cos\left(\frac{jx^4-kx^5}{R}+\theta\right)+\cos\left(\frac{kx^4+jx^5}{R}-\theta\right)\right]. \end{aligned} \tag{2.30}$$

Next, we impose the other two boundary conditions, which relate the $x^4 = L$ and $x^5 = L$ boundaries, and the constraint (2.17). Applying the boundary condition for Φ, eq. (2.13), to the solution (2.30), we find an equation that has to hold for any $y \in [0, L]$ and arbitrary j and k. Therefore, the coefficients of $\cos ky/R$, $\sin ky/R$, $\cos jy/R$ and $\sin jy/R$ must vanish independently:

$$\begin{aligned} \left(C^+_1 + C^+_2\right)\left(e^{i(\tilde\theta+\theta)} - 1\right)\cos(j\pi+\theta) &= 0\,, \\ \left(C^+_1 - C^+_2\right)\left(e^{i(\tilde\theta+\theta)} + 1\right)\sin(j\pi+\theta) &= 0\,, \\ \left(C^+_1 + e^{2i\theta}C^+_2\right)\left(e^{i(\tilde\theta+\theta)} - 1\right)\cos(k\pi+\theta) &= 0\,, \\ \left(C^+_1 - e^{2i\theta}C^+_2\right)\left(e^{i(\tilde\theta+\theta)} + 1\right)\sin(k\pi+\theta) &= 0\,. \end{aligned} \tag{2.31}$$

Here we have used the restriction on θ, eq. (2.28). Following the same procedure, the boundary condition for the derivatives of Φ, eq. (2.14), leads to four more equations which can be obtained from eqs. (2.31) by substituting $\tilde\theta'$ for $\tilde\theta$ and interchanging cos and sin. Thus, there are eight equations altogether. We have to find a solution to this set of equations with at least one of C^+_1 and C^+_2 being nonzero, and which is subject to the constraint (2.17). This is possible only if $\tilde\theta$, $\tilde\theta'$, j and k satisfy certain conditions. For

$$e^{i\tilde\theta'} = -e^{i\tilde\theta} \tag{2.32}$$

we obtain that either

$$e^{i\tilde\theta} = e^{i\theta} \qquad \text{and} \qquad j,k \in \mathbb{Z}\,, \tag{2.33}$$

or

$$e^{i\tilde\theta} = -e^{i\theta} \qquad \text{and} \qquad j+\frac{1}{2}\,,\; k+\frac{1}{2} \in \mathbb{Z}\,. \tag{2.34}$$

For $f(y,L)\,\partial_5 f|_{(x^4,x^5)=(y,L)} = 0$, the solutions are again just a subset of the solutions obtained when eq. (2.33) is satisfied, with $e^{2i\theta} = 1$ and $C^\pm_2 = \pm C^\pm_1$.

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The conclusion so far is that the most general folding boundary conditions that allow a nontrivial solution to the six-dimensional Klein-Gordon equation are given by

$$\begin{aligned}
\Phi(x^\mu, y, 0) &= e^{in\pi/2}\Phi(x^\mu, 0, y)\,, \\
\Phi(x^\mu, y, L) &= (-1)^l e^{in\pi/2}\Phi(x^\mu, L, y)\,, \\
\partial_5\Phi|_{(x^4,x^5)=(y,0)} &= -e^{in\pi/2}\,\partial_4\Phi|_{(x^4,x^5)=(0,y)}\,, \\
\partial_5\Phi|_{(x^4,x^5)=(y,L)} &= -(-1)^l e^{in\pi/2}\,\partial_4\Phi|_{(x^4,x^5)=(L,y)}\,,
\end{aligned} \tag{2.35}$$

where l, n are integers that can be restricted to $n = 0, 1, 2, 3$ and $l = 0, 1$. It is interesting that the folding boundary conditions do not depend on a continuous parameter, but rather there are only eight self-consistent choices.

Although two coefficients, $C^+_{1,2}$, remain unknown in the solution (2.30), they multiply two functions of x^4 and x^5 that either differ only by an interchange of j and k and a factor of $e^{-i\theta}$, or are identical when $k = 0$. It turns out that we can keep only one of these two coefficients, and still form a complete set of functions on the square, which is the necessary and sufficient condition for having a general Fourier decomposition as in eq. (2.19). Furthermore, the normalization condition,

$$\frac{1}{L^2}\int_0^L dx^4 \int_0^L dx^5 \left[f^{(j,k)}(x^4,x^5)\right]^* f^{(j',k')}(x^4,x^5) = \delta_{j,j'}\,\delta_{k,k'}\,, \tag{2.36}$$

determines the last coefficient up to a phase factor which we choose to be one. Explicitly, the solutions to eq. (2.21) can be written as

$$f_n^{(j,k)}(x^4,x^5) = \frac{1}{1+\delta_{j,0}\delta_{k,0}}\left[e^{-in\pi/2}\cos\pi\left(\frac{jx^4+kx^5}{L}+\frac{n}{2}\right) + \cos\pi\left(\frac{kx^4-jx^5}{L}+\frac{n}{2}\right)\right], \tag{2.37}$$

with $j + l/2$ and $k + l/2$ integers.

The functions $f_n^{(j,k)}$ form a complete orthonormal set on the square if

$$\frac{1}{L^2}\sum_{j,k}\left[f_n^{(j,k)}(x^4,x^5)\right]^* f_n^{(j,k)}(x'^4,x'^5) = \delta(x'^4-x^4)\,\delta(x'^5-x^5)\,. \tag{2.38}$$

The allowed values for j and k must be chosen such that the above completeness condition is satisfied. In practice it is easier to observe first that

$$f_n^{(-j,k)} = (-1)^n f_n^{(j,-k)} = e^{in\pi/2} f_n^{(k,j)}\,, \tag{2.39}$$

so that it is sufficient to take $j > 0$, $k \geq 0$ and $j = k = 0$.

Thus, for the boundary conditions with $l = 0$, j and k take all integer values with

$$j \geq 1 - \delta_{n,0}\delta_{k,0}\,, \qquad k \geq 0\,. \tag{2.40}$$

One can check that the completeness condition eq. (2.38) is then satisfied. We have obtained a tower of four-dimensional fields $\Phi^{(j,k)}$ labeled by two integers, with masses

$$M^{(j,k)} = \sqrt{M_0^2 + \frac{j^2+k^2}{R^2}}\,. \tag{2.41}$$

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(j,k)	(1,0)	(1,1)	(2,0)	$(2,1)$ $(1,2)$	(2,2)	(3,0)	$(3,1)$ $(1,3)$	$(3,2)$ $(2,3)$	(4,0)
$M_{j,k}R$	1	$\sqrt{2}$	2	$\sqrt{5}$	$2\sqrt{2}$	3	$\sqrt{10}$	$\sqrt{13}$	4
(j,k)	$(4,1)$ $(1,4)$	(3,3)	$(4,2)$ $(2,4)$	$(4,3)$ $(3,4)$ $(5,0)$	$(5,1)$ $(1,5)$	$(5,2)$ $(2,5)$	(4,4)	$(5,3)$ $(3,5)$	(6,0)
$M_{j,k}R$	$\sqrt{17}$	$3\sqrt{2}$	$2\sqrt{5}$	5	$\sqrt{26}$	$\sqrt{29}$	$4\sqrt{2}$	$\sqrt{34}$	6

Table 1: KK modes with $M_{j,k} \leq 6M_{1,0}$, obtained for folding boundary conditions with $l = 0$.

(j,k)	$(\frac{1}{2},\frac{1}{2})$	$(\frac{3}{2},\frac{1}{2})$ $(\frac{1}{2},\frac{3}{2})$	$(\frac{3}{2},\frac{3}{2})$	$(\frac{5}{2},\frac{1}{2})$ $(\frac{1}{2},\frac{5}{2})$	$(\frac{5}{2},\frac{3}{2})$ $(\frac{3}{2},\frac{5}{2})$	$(\frac{5}{2},\frac{5}{2})$ $(\frac{7}{2},\frac{1}{2})$ $(\frac{1}{2},\frac{7}{2})$	$(\frac{7}{2},\frac{3}{2})$ $(\frac{3}{2},\frac{7}{2})$
$\sqrt{2}M_{j,k}R$	1	$\sqrt{5}$	3	$\sqrt{13}$	$\sqrt{17}$	5	$\sqrt{29}$

Table 2: KK modes with $M_{j,k} \leq 6M_{\frac{1}{2},\frac{1}{2}}$, obtained for folding boundary conditions with $l = 1$.

As usual, we will refer to these fields as KK modes. The KK numbers, j and k, of the lightest KK modes are shown in table 1. For $n = 0$, there is an additional state: $j = k = 0$. This is a state of zero momentum ("zero mode") along both compact dimensions.

For the boundary conditions with $l = 1$, j and k take all half-integer values satisfying

$$j \geq \frac{1}{2}, \qquad k \geq \frac{1}{2}. \tag{2.42}$$

In table 2 are listed the KK numbers and masses of the lightest KK modes in this case.

All the results obtained in this section for a complex scalar with $n = 0$ or $n = 2$ apply to the case of a real scalar field as well (note that $f_n^{(j,k)}$ are complex functions for $n = 1, 3$).

3. Fermions on a square: chiral boundary conditions

We now turn to free spin-1/2 fields in six dimensions. The Clifford algebra is generated by six anti-commuting matrices: Γ^α, $\alpha = 0, 1, \ldots, 5$. The minimal dimensionality of these matrices is 8×8. The Γ matrices can be used to construct a spinor representation of the SO(1,5) Lorentz symmetry, with the generators explicitly given by

$$\frac{\Sigma^{\alpha\beta}}{2} = \frac{i}{4}[\Gamma^\alpha, \Gamma^\beta]. \tag{3.1}$$

This Lorentz representation is reducible and contains two irreducible Weyl representations, which have different eigenvalues of the chirality operator. The two six-dimensional chiralities, labeled by $+$ and $-$, are projected by the operators

$$P_\pm = \frac{1}{2}\left(1 \pm \bar{\Gamma}\right), \tag{3.2}$$

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where the six-dimensional chirality operator

$$\begin{aligned}\bar{\Gamma} &= \frac{1}{6!}\epsilon_{\alpha_0\alpha_1\cdots\alpha_5}\Gamma^{\alpha_0}\Gamma^{\alpha_1}\cdots\Gamma^{\alpha_5} \\ &= \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\end{aligned} \tag{3.3}$$

is a self-adjoint matrix that anticommutes with all Γ^α's. The chiral fermions in six dimensions have four components.

Upon compactification in the x^4, x^5 plane, the SO(1,3) Lorentz symmetry generated by $\Sigma^{\mu\nu}/2$, $\mu,\nu = 0,1,2,3$, remains unbroken. There are two chiralities under SO(1,3), labeled as usual by L and R. These are projected by

$$P_{L,R} = \frac{1}{2}\left(1 \pm i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\right). \tag{3.4}$$

A six-dimensional chiral fermion, $\Psi_\pm \equiv P_\pm\Psi$, decomposes into two fermions of definite chirality under SO(1,3):

$$\Psi_\pm(x^\mu, x^4, x^5) = \Psi_{\pm L}(x^\mu, x^4, x^5) + \Psi_{\pm R}(x^\mu, x^4, x^5)\,, \tag{3.5}$$

where

$$\Psi_{\pm_{L,R}} \equiv P_{L,R}P_\pm\Psi\,. \tag{3.6}$$

As in section 2, we consider the compactification on a square: $0 \le x^4, x^5 \le L$. For definiteness, we analyze the case of a chirality + fermion. At the end of this section we briefly comment on the differences for the − chirality. The action for a free six-dimensional chiral fermion is

$$S_\Psi = \int d^4x \int_0^L dx^4 \int_0^L dx^5 \frac{i}{2}\left[\overline{\Psi}_+\Gamma^\alpha\partial_\alpha\Psi_+ - \left(\partial_\alpha\overline{\Psi}_+\right)\Gamma^\alpha\Psi_+\right]. \tag{3.7}$$

Under an arbitrary variation of the field, $\delta\Psi_+(x^\mu, x^4, x^5)$, the action has to be stationary both inside the square and on its boundary:

$$\begin{aligned}\delta S^v_\Psi &= -\int d^4x \int_0^L dx^4 \int_0^L dx^5 i\left(\partial_\alpha\overline{\Psi}_+\right)\Gamma^\alpha\delta\Psi_+ = 0\,, \\ \delta S^s_\Psi &= \frac{i}{2}\int d^4x\Bigg[\int_0^L dx^4\left(\overline{\Psi}_+\Gamma^5\delta\Psi_+\big|_{x^5=L} - \overline{\Psi}_+\Gamma^5\delta\Psi_+\big|_{x^5=0}\right) + \\ &\qquad + \int_0^L dx^5\left(\overline{\Psi}_+\Gamma^4\delta\Psi_+\big|_{x^4=L} - \overline{\Psi}_+\Gamma^4\delta\Psi_+\big|_{x^4=0}\right)\Bigg] = 0\,.\end{aligned} \tag{3.8}$$

The first equation implies that Ψ_+ is a solution to the six-dimensional Weyl equation, which can be decomposed into two equations:

$$\begin{aligned}\Gamma^\mu\partial_\mu\Psi_{+L} &= -\left(\Gamma^4\partial_4 + \Gamma^5\partial_5\right)\Psi_{+R}\,, \\ \Gamma^\mu\partial_\mu\Psi_{+R} &= -\left(\Gamma^4\partial_4 + \Gamma^5\partial_5\right)\Psi_{+L}\,.\end{aligned} \tag{3.9}$$

The second equation (3.8) restricts the values of Ψ_+ on the boundary:

$$\begin{aligned}
&\overline{\Psi}_+\Gamma^4\,\delta\Psi_+\big|_{(x^4,x^5)=(L,y)} + \overline{\Psi}_+\Gamma^5\,\delta\Psi_+\big|_{(x^4,x^5)=(y,L)} - \\
&- \overline{\Psi}_+\Gamma^4\,\delta\Psi_+\big|_{(x^4,x^5)=(0,y)} - \overline{\Psi}_+\Gamma^5\,\delta\Psi_+\big|_{(x^4,x^5)=(y,0)} = 0\,,
\end{aligned} \tag{3.10}$$

for any $y \in [0, L]$.

We consider "folding" boundary conditions analogous to those imposed on the scalar field in section 2. The important feature of these boundary conditions is that they distinguish the four-dimensional chiralities. Explicitly, the phases that relate the fields on adjacent sides of the square are different for left- and right-handed fermions:

$$\begin{aligned}
\Psi_{+_{L,R}}(x^\mu, y, 0) &= e^{i\theta_{L,R}}\Psi_{+_{L,R}}(x^\mu, 0, y)\,, \\
\Psi_{+_{L,R}}(x^\mu, y, L) &= e^{i\tilde{\theta}_{L,R}}\Psi_{+_{L,R}}(x^\mu, L, y)\,,
\end{aligned} \tag{3.11}$$

for any $y \in [0, L]$. As in the case of scalars, the above equations imply that

$$\begin{aligned}
\partial_4\Psi_{+_{L,R}}\big|_{(x^4,x^5)=(y,0)} &= e^{i\theta_{L,R}}\;\partial_5\Psi_{+_{L,R}}\big|_{(x^4,x^5)=(0,y)}\,, \\
\partial_4\Psi_{+_{L,R}}\big|_{(x^4,x^5)=(y,L)} &= e^{i\tilde{\theta}_{L,R}}\;\partial_5\Psi_{+_{L,R}}\big|_{(x^4,x^5)=(L,y)}\,.
\end{aligned} \tag{3.12}$$

The study of complex scalars presented in section 2 has shown that additional boundary conditions, eqs. (2.12), must be imposed on the field derivatives in order to have the same lagrangians at the identified points. We now show that in the case of fermions the boundary conditions eqs. (3.11) are sufficient to ensure the equality of the lagrangians at the identified points. To this end, we write the first equation of motion (3.9) at $(x^4, x^5) = (0, y)$, multiply it by $e^{i\theta_L}$, and subtract the same equation evaluated at $(x^4, x^5) = (y, 0)$. Using then eqs. (3.12) and the identities

$$\begin{aligned}
\Gamma^4 P_L P_\pm &= \mp i\Gamma^5 P_L P_\pm\,, \\
\Gamma^4 P_R P_\pm &= \pm i\Gamma^5 P_R P_\pm\,,
\end{aligned} \tag{3.13}$$

we obtain

$$\partial_4\Psi_{+R}\big|_{(x^4,x^5)=(0,y)} + ie^{-i\theta_L}\;\partial_5\Psi_{+R}\big|_{(x^4,x^5)=(y,0)} = i\left[1 - ie^{i(\theta_R-\theta_L)}\right]\partial_5\Psi_{+R}\big|_{(x^4,x^5)=(0,y)}\,. \tag{3.14}$$

Writing the second equation of motion (3.9) at $(x^4, x^5) = (0, y)$, multiplying it by $e^{i\theta_R}$, and subtracting the same equation evaluated at $(x^4, x^5) = (y, 0)$ gives

$$\partial_4\Psi_{+L}\big|_{(x^4,x^5)=(0,y)} - ie^{-i\theta_R}\;\partial_5\Psi_{+L}\big|_{(x^4,x^5)=(y,0)} = i\left[1 + ie^{-i(\theta_R-\theta_L)}\right]\partial_5\Psi_{+L}\big|_{(x^4,x^5)=(0,y)}\,. \tag{3.15}$$

Based on the above two equations, one can check that the lagrangian at $(x^4, x^5) = (y, 0)$ is equal to the one at $(x^4, x^5) = (0, y)$.

The same procedure gives two equations for the derivatives at $(x^4, x^5) = (L, y)$ and (y, L) which are analogous to eqs. (3.14) and (3.15), except for the replacement of $\theta_{L,R}$ by $\tilde{\theta}_{L,R}$. As a result, the lagrangians at $(x^4, x^5) = (L, y)$ and (y, L) are also equal.

The boundary conditions (3.11) must satisfy the stationarity condition (3.10). Given that the field variations on opposite sides of the square are independent, and that the variations of Ψ_{+L} and Ψ_{+R} are independent at any point, eqs. (3.10) are satisfied if and only if

$$\begin{aligned}\left[1 - ie^{i(\theta_R-\theta_L)}\right] \overline{\Psi}_{+L}(x^\mu,0,y)\Gamma^4\Psi_{+R}(x^\mu,0,y) &= 0\,,\\ \left[1 - ie^{i(\tilde{\theta}_R-\tilde{\theta}_L)}\right] \overline{\Psi}_{+L}(x^\mu,L,y)\Gamma^4\Psi_{+R}(x^\mu,L,y) &= 0\,.\end{aligned} \tag{3.16}$$

We now show that eqs. (3.11), (3.14), (3.15) and (3.16) imply that both Ψ_{+L} and Ψ_{+R} satisfy the same boundary conditions as in the scalar case of section 2. To see this, note first that for

$$e^{i(\theta_L-\theta_R)} = i \tag{3.17}$$

eqs. (3.14) and (3.15) take the form

$$\partial_4\Psi_{+L,R}\big|_{(x^4,x^5)=(0,y)} = -e^{-i\theta_{L,R}}\ \partial_5\Psi_{+L,R}\big|_{(x^4,x^5)=(y,0)}\,, \tag{3.18}$$

which is the same as the boundary condition for the scalar derivatives, eq. (2.12), with the identification $e^{i\theta'} = -e^{i\theta_{L,R}}$. Therefore, in this case we automatically obtain the restriction given by the relation (2.27) between θ and θ'.

If eq. (3.17) is not assumed, then eq. (3.16) requires that either Ψ_{+R} or Ψ_{+L} vanish at the $(0,y)$ points. These two cases yield the same result, so we present here only the $\Psi_{+R}(0,y) = 0$ case. Then, $\Psi_{+R}(y,0) = 0$ due to the folding boundary condition eq. (3.11). The six-dimensional Dirac equation (3.9), together with the identity eq. (3.13), implies that

$$(\partial_4 + i\partial_5)\Psi_{+L}|_{(x^4,x^5)=(y,0)} = (\partial_4 + i\partial_5)\Psi_{+L}|_{(x^4,x^5)=(0,y)} = 0\,. \tag{3.19}$$

These two equations together with the folding boundary condition for Ψ_{+L} lead again to eq. (3.18) for Ψ_{+L}. Using eq. (3.18) to eliminate $\partial_5\Psi_{+L}$ at $(y,0)$ in favor of $\partial_4\Psi_{+L}$ at $(0,y)$ on the left-hand side of eq. (3.15), then using eq. (3.19) to eliminate $\partial_5\Psi_{+L}$ at $(0,y)$ in favor of $\partial_4\Psi_{+L}$ at $(0,y)$ on its right-hand side, and observing that there are always values of y such that $\partial_4\Psi_{+L}|_{(0,y)} \neq 0$, we recover eq. (3.17). Hence, the boundary condition (3.18) for Ψ_{+R} is also satisfied.

Similar conclusions (with $\theta_{L,R}$ replaced by $\tilde{\theta}_{L,R}$) apply at (L,y) and (y,L):

$$e^{i(\tilde{\theta}_L-\tilde{\theta}_R)} = i\,, \tag{3.20}$$

and the boundary conditions are those encountered in the scalar case.

We can now solve the six-dimensional Weyl equation. First we observe that the coupled first-order equations (3.9) imply that both Ψ_{+L} and Ψ_{+R} obey the six-dimensional Klein-Gordon equation for a massless field. Therefore, similarly to the case of a scalar discussed in section 2, Ψ_{+L} and Ψ_{+R} can be decomposed in Fourier modes. However, further complications arise because the Weyl fermions in six dimensions have four components, while in four dimensions they have two components. The Γ matrices can be written as the direct product of the usual four-dimensional γ matrices with the Pauli matrices:

$$\Gamma^\mu = \gamma^\mu \otimes \sigma^0\,, \qquad \Gamma^{4,5} = i\gamma_5 \otimes \sigma^{1,2}\,, \tag{3.21}$$

where σ^0 is the 2×2 unit matrix. Note that $\overline{\Gamma} = \gamma_5 \otimes \sigma^3$. The KK decomposition can then be written as follows:

$$\begin{aligned}
\Psi_{+L}(x^\mu, x^4, x^5) &= \frac{1}{L}\sum_{j,k} f_{+L}^{(j,k)}(x^4, x^5)\,\Psi_{+L}^{(j,k)}(x^\mu) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
\Psi_{+R}(x^\mu, x^4, x^5) &= \frac{1}{L}\sum_{j,k} f_{+R}^{(j,k)}(x^4, x^5)\,\Psi_{+R}^{(j,k)}(x^\mu) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (3.22)
\end{aligned}$$

where $f_{+L,R}^{(j,k)}$ are scalar functions of x^4 and x^5, while $\Psi_{+L,R}^{(j,k)}$ are Weyl fermions in four dimensions. The γ matrices act on $\Psi_{+L,R}^{(j,k)}$, and the Pauli matrices act on the 2-columns shown in the above equation. The $\Psi_{+L}^{(j,k)}$ and $\Psi_{+R}^{(j,k)}$ fields form a Dirac fermion in four dimensions

$$(i\gamma^\mu \partial_\mu - M_{j,k})\left(\Psi_{+L}^{(j,k)} + \Psi_{+R}^{(j,k)}\right) = 0\,, \qquad (3.23)$$

which explains why the real numbers j and k are the same for the KK decompositions of both $\Psi_{+L}^{(j,k)}$ and $\Psi_{+R}^{(j,k)}$ shown in eq. (3.22). The mass $M_{j,k}$ turns out to be the same as in the scalar case, i.e., it is given by eq. (2.23).

Inserting the KK decomposition in the first-order equations (3.9) we obtain that $f_{+L,R}^{(j,k)}$ must be solutions to

$$\begin{aligned}
(\partial_4 - i\partial_5)\, f_{+R}^{(j,k)} &= M_{j,k} f_{+L}^{(j,k)}\,, \\
(\partial_4 + i\partial_5)\, f_{+L}^{(j,k)} &= -M_{j,k} f_{+R}^{(j,k)}\,. \qquad (3.24)
\end{aligned}$$

Acting on the first equation with $\partial_4 + i\partial_5$ and then using the second equation we find that f_{+R} satisfies the second-order equation encountered in the scalar case, eq. (2.21). Having also the same boundary conditions as for the scalars, as we discussed above, implies that $f_{+R}^{(j,k)}$ is given by the right-hand side of eq. (2.37). Therefore, $f_{+R}^{(j,k)}$ depends on an integer that can take four values, $n_R^+ = 0, 1, 2, 3$, and for each of these cases there is a solution where j and k are integers (a case labeled by $l^+ = 0$), and a different solution ($l^+ = 1$) where they are half-integers. Given one of these solutions for $f_{+R}^{(j,k)}$, the first equation in (3.24) determines $f_{+L}^{(j,k)}$ and gives a solution of the same form except for a shift by one in n_R^+ and an overall phase factor:

$$\begin{aligned}
f_{+R}^{(j,k)} &= f_{n_R^+}^{(j,k)}\,, \\
f_{+L}^{(j,k)} &= \frac{k + ij}{\sqrt{j^2 + k^2}}\, f_{1+n_R^+}^{(j,k)}\,, \qquad (3.25)
\end{aligned}$$

where $f_n^{(j,k)}$ is given by eq. (2.37).

To summarize, a chirality + fermion with an expansion in terms of the above KK wavefunctions, as in eq. (3.22), satisfies the boundary conditions

$$\begin{aligned}
\Psi_{+R}(x^\mu, y, 0) &= e^{in_R^+\pi/2}\,\Psi_{+R}(x^\mu, 0, y)\,, \\
\Psi_{+R}(x^\mu, y, L) &= (-1)^{l^+} e^{in_R^+\pi/2}\,\Psi_{+R}(x^\mu, L, y)\,, \\
\partial_5 \Psi_{+R}\big|_{(x^4,x^5)=(y,0)} &= -e^{in_R^+\pi/2}\;\partial_4 \Psi_{+R}\big|_{(x^4,x^5)=(0,y)}\,, \\
\partial_5 \Psi_{+R}\big|_{(x^4,x^5)=(y,L)} &= -(-1)^{l^+} e^{in_R^+\pi/2}\;\partial_4 \Psi_{+R}\big|_{(x^4,x^5)=(L,y)}\,, \qquad (3.26)
\end{aligned}$$

for the right-handed component, while for the left-handed component the same boundary conditions apply, except for n_R^+ being replaced by

$$n_L^+ = n_R^+ + 1 \bmod 4\,. \tag{3.27}$$

The $-$ chirality can be treated in an analogous fashion. The only difference compared to the $+$ chirality discussed above arises from eq. (3.13), and leads to the interchange of L and R in the equations following eq. (3.13). Thus, Ψ_{-_L} and Ψ_{-_R} have folding boundary conditions characterized by two integers each, n_L^-, l^- and n_R^-, l^-, respectively, with $n_L^-, n_R^- = 0, 1, 2, 3$, $l^- = 0, 1$, and

$$n_L^- = n_R^- - 1 \bmod 4\,. \tag{3.28}$$

Their KK wave functions are related as follows:

$$\begin{aligned} f_{-_L}^{(j,k)} &= f_{n_L^-}^{(j,k)}\,, \\ f_{-_R}^{(j,k)} &= \frac{k + ij}{\sqrt{j^2 + k^2}}\, f_{1+n_L^-}^{(j,k)}\,, \end{aligned} \tag{3.29}$$

where β is some arbitrary phase.

The KK spectrum of a six-dimensional chiral fermion consists of four-dimensional vector-like fermions of masses M_{jk}, as in tables 1 and 2. Only the zero-modes are four-dimensional chiral fermions. It is interesting that there are two kinds of zero-modes for each four-dimensional chirality: depending on whether the six-dimensional fermion has chirality $+$ or $-$, the tower of KK modes that includes a left-handed zero-mode is paired with a tower of KK modes with wavefunctions of the f_3 or f_1 type, and vice-versa for a right-handed zero-mode.

The discussion in this section has been restricted so far to chiral six-dimensional fermions. It is also useful to analyze the case of a vector-like six-dimensional fermion, Ψ, of mass M_0. The boundary conditions and solutions to the field equations for Ψ_+ and Ψ_-, obtained independently above, continue to apply when both chiralities are present, provided the equality of lagrangians at identified points is satisfied. Therefore, the presence in the lagrangian of the mass term

$$-M_0 \overline{\Psi}\Psi = -M_0 \left(\overline{\Psi}_{+_L}\Psi_{-_R} + \overline{\Psi}_{+_R}\Psi_{-_L}\right) + \text{h.c.} \tag{3.30}$$

relates the KK wave functions of the Ψ_+ and Ψ_- components of Ψ. The mass term is the same at identified boundary points when $n_L^\pm = n_R^\mp$. Another implication of the six-dimensional mass term is that the $M_{j,k}$ mass of a KK mode is now replaced by a matrix:

$$\left(\overline{\Psi}_{+_L}^{(j,k)}\,, \overline{\Psi}_{-_L}^{(j,k)}\right) \begin{pmatrix} M_{j,k} & M_0 \\ M_0^* & -M_{j,k} \end{pmatrix} \begin{pmatrix} \Psi_{+_R}^{(j,k)} \\ \Psi_{-_R}^{(j,k)} \end{pmatrix}. \tag{3.31}$$

Both eigenvalues are equal to $(M_0^2 + M_{j,k}^2)^{1/2}$, which is the same as the mass of the (j,k) mode of a scalar with bulk mass M_0. Thus, a vector-like six-dimensional fermion includes two degenerate towers of vector-like KK modes, and at most a single vector-like $j = k = 0$ state of mass M_0 (when $n_L^+ n_L^- = 0$). Note that this is also the case for the T^2/Z_2 orbifold [16], and that the degeneracy is lifted by loops if Ψ_+ and Ψ_- have different interactions [22].

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4. Symmetries

So far we have analyzed free scalar and fermion fields, and found that the equations of motion have nontrivial solutions only if the phases associated with the folding boundary conditions are restricted to a discrete set of values. In this section we show that those restrictions lead to the existence of certain symmetries that are obeyed in theories with any number of scalar and fermion fields, and with any type of local six-dimensional Lorentz-invariant interactions.

4.1 $Z_8 \times Z_2^{\rm AP}$ invariance

Consider a theory with a number of six-dimensional complex scalar fields, $\Phi_i(x^\mu, x^4, x^5)$ with $i = 1, \ldots, p$. By studying the free part of the lagrangian, as we did above, each of these is subject to folding boundary conditions as in eq. (2.35), but the integers $n_i = 0, 1, 2, 3$ and $l_i = 0, 1$ that determine the boundary conditions may differ for different fields, and are therefore labeled by a flavor index i.[1] The most general interactions involving these fields which do not involve derivatives are of the type

$$\prod_{i=1}^{p} (\Phi_i)^{m_i} (\Phi_i^\dagger)^{m'_i} \,, \tag{4.1}$$

where $m_i, m'_i \geq 0$ are integers. The key fact, which follows from the form of the folding boundary conditions, eq. (2.35), is that equality of the lagrangians at the identified boundary points $(y, 0)$ and $(0, y)$ requires that the overall phase difference is a multiple of 2π:

$$\sum_{i=1}^{p} n_i (m_i - m'_i) = 0 \bmod 4 \,. \tag{4.2}$$

This equation implies that all such interaction terms are invariant under the Z_4 transformations

$$\Phi_i(x^\mu, x^4, x^5) \mapsto e^{-i n_i \pi/2} \Phi_i(x^\mu, x^4, x^5) \,. \tag{4.3}$$

Furthermore, equality of the lagrangians at the identified boundary points (y, L) and (L, y) requires in addition

$$\sum_{i=1}^{p} l_i (m_i - m'_i) = 0 \bmod 2 \,, \tag{4.4}$$

so that the operators (4.1) are invariant under the additional Z_2 transformations

$$\Phi_i(x^\mu, x^4, x^5) \mapsto (-1)^{l_i} \Phi_i(x^\mu, x^4, x^5) \,. \tag{4.5}$$

We will refer to this Z_2 symmetry as $Z_2^{\rm AP}$.

[1] If the Φ_i span a representation of a non-abelian internal symmetry, it may be possible to impose eqs. (2.9) and (2.13) with $\Phi = (\Phi_1, \ldots, \Phi_p)$ and the phase $e^{i\theta}$ replaced by a matrix. This may be interesting as a higher dimensional mechanism for symmetry breaking [23], but we do not consider this possibility here.

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The six-dimensional Lorentz invariance ensures that the operators that include derivatives in the most general way are also invariant under the above Z_4 and $Z_2^{\rm AP}$ transformations. To see this, recall first that only the derivatives of a field along the compact dimensions have boundary conditions with an n integer different than for the field itself:

$$\left(\partial_4 \pm i\partial_5\right) \Phi_i\big|_{(x^4,x^5)=(y,0)} = e^{-i(n_i \mp 1)\pi/2} \left(\partial_4 \pm i\partial_5\right) \Phi_i\big|_{(x^4,x^5)=(0,y)} . \tag{4.6}$$

Six-dimensional Lorentz invariance allows only two types of combinations of derivatives,

$$\begin{aligned} \partial^\alpha\Phi_1\partial_\alpha\Phi_2 &= \partial^\mu\Phi_1\partial_\mu\Phi_2 + \frac{1}{2}[\left(\partial_4 + i\partial_5\right)\Phi_1 \left(\partial_4 - i\partial_5\right)\Phi_2 + \\ &\qquad\qquad + \left(\partial_4 - i\partial_5\right)\Phi_1 \left(\partial_4 + i\partial_5\right)\Phi_2] \\ \epsilon^{\alpha_1\cdots\alpha_6}\partial_{\alpha_1}\Phi_1\cdots\partial_{\alpha_6}\Phi_6 &= \frac{i}{2}\sum \epsilon^{\mu_1\cdots\mu_4}\partial_{\mu_1}\Phi_{i_1}\cdots\partial_{\mu_4}\Phi_{i_4}\times \\ &\quad \times\left[\left(\partial_4 + i\partial_5\right)\Phi_{i_5}\left(\partial_4 - i\partial_5\right)\Phi_{i_6} - \left(\partial_4 - i\partial_5\right)\Phi_{i_5}\left(\partial_4 + i\partial_5\right)\Phi_{i_6}\right], \end{aligned} \tag{4.7}$$

where the sum in the right-hand side of the second equation is over the permutations of the set of indices $\{i_1,\ldots,i_6\} = \{1,\ldots,6\}$. From eq. (4.6) then follows that even in the presence of derivatives the equality of the lagrangians at the identified boundary points requires that eqs. (4.2) and (4.4) be satisfied. Therefore, the scalar action is invariant under a $Z_4 \times Z_2^{\rm AP}$ symmetry defined by eqs. (4.3) and (4.5). We will refer to the n_i and l_i that characterize the folding boundary conditions of Φ_i as the $Z_4 \times Z_2^{\rm AP}$ charges of Φ_i.

In the case of fermions, the same results clearly apply to operators where all the derivatives have Lorentz indices contracted among themselves. However, the gamma matrices also carry Lorentz indices, and when these are contracted with the indices of the derivatives the equality of the lagrangians at $(y,0)$ and $(0,y)$ leads to additional constraints. For example, a kinetic term $i\overline{\Psi}{}^i_+\Gamma^\alpha\partial_\alpha\Psi^i_+$, where the upper index i labels different flavors, includes a piece

$$i\overline{\Psi}{}^i_{+L}\left(\Gamma^4 + i\Gamma^5\right)\left(\partial_4 - i\partial_5\right)\Psi^i_{+R} \tag{4.8}$$

which is consistent with the folding boundary conditions only if $n^+_{Ri} - n^+_{Li} = -1$ [see eq. (3.26)]. Thus, for fermions, the naive Z_4 transformation eq. (4.3) is not a symmetry.

However, the fermion kinetic term is invariant under the Z_8 transformations

$$\begin{aligned} \Psi^i_{\pm R}(x^\mu,x^4,x^5) &\mapsto e^{-i(\pm 1/2 + n^\pm_{Ri})\pi/2}\Psi^i_{\pm R}(x^\mu,x^4,x^5), \\ \Psi^i_{\pm L}(x^\mu,x^4,x^5) &\mapsto e^{-i(\mp 1/2 + n^\pm_{Li})\pi/2}\Psi^i_{\pm L}(x^\mu,x^4,x^5), \end{aligned} \tag{4.9}$$

where we also included the transformations for the $-$ chirality. To see that the above transformation is a symmetry of a theory with any interactions, we first note that it can be written as

$$\Psi^i(x^\mu,x^4,x^5) \mapsto \Psi'^i(x^\mu,x^4,x^5) \equiv e^{-i(\pi/2)\Sigma_{45}/2}\Psi^i(x^\mu,-x^5,x^4), \tag{4.10}$$

where Ψ^i is a six-dimensional spinor and we used the fact that the fermion fields on the folded square satisfy

$$\begin{aligned} \Psi^i_{\pm R}(x^\mu,-x^5,x^4) &= e^{-in^\pm_{Ri}\pi/2}\Psi^i_{\pm R}(x^\mu,x^4,x^5), \\ \Psi^i_{\pm L}(x^\mu,-x^5,x^4) &= e^{-in^\pm_{Li}\pi/2}\Psi^i_{\pm L}(x^\mu,x^4,x^5), \end{aligned} \tag{4.11}$$

which follow from the corresponding property of the KK wavefunctions in eq. (2.37),

$$f_n^{(j,k)}(-x^5, x^4) = e^{-in\pi/2} f_n^{(j,k)}(x^4, x^5)\,. \tag{4.12}$$

It is now clear that the transformations (4.9) correspond to a rotation by $\pi/2$ in the plane of the compact dimensions around the point $(x^4, x^5) = (0, 0)$, given that the fields initially defined on the square $0 < x^4, x^5 < L$ can be analytically continued to the whole plane.

All *local* operators in the (compactified) six-dimensional theory are restricted by the six-dimensional Lorentz symmetry and in particular by the transformations (4.10). More precisely, such operators satisfy

$$\mathcal{O}\left[\overline{\Psi'}^{i_1}(x^\alpha), \Psi'^{i_2}(x^\alpha), \frac{\partial}{\partial x^\alpha}\right] = \mathcal{O}\left[\overline{\Psi}^{i_1}(x'^\alpha), \Psi^{i_2}(x'^\alpha), \frac{\partial}{\partial x'^\alpha}\right] \equiv \mathcal{O}(x') \tag{4.13}$$

where $\Psi'^i(x^\alpha)$ are the Lorentz transformed fields. Equation (4.13) is the statement that the local operators that may appear in the six-dimensional lagrangian are Lorentz scalars. The six-dimensional Lorentz symmetry is broken by compactification, but this does not relax the restriction (4.13) on local operators. For the transformation (4.10), the left-hand side in eq. (4.13) can be written as

$$\mathcal{O}\left[\overline{\Psi'}^{i_1}_{\pm R,L}(x^\alpha), \Psi'^{i_2}_{\pm R,L}(x^\alpha), \frac{\partial}{\partial x^\alpha}\right] = e^{i(\pi/2)\left(\sum_{i_2} q_{i_2} - \sum_{i_1} q_{i_1}\right)}\mathcal{O}(x)\,, \tag{4.14}$$

where $q_i = \pm 1/2 + n_i$ are the appropriate charges as defined in eq. (4.9). In addition, the equality of the lagrangians at $(0, y)$ and $(y, 0)$ implies that

$$\mathcal{O}(x')\big|_{(x'^4, x'^5)=(0,y)} = \mathcal{O}(x)\big|_{(x^4,x^5)=(y,0)}\,, \tag{4.15}$$

which together with eqs. (4.13) and (4.14) requires

$$\sum_{i_2} q_{i_2} - \sum_{i_1} q_{i_1} = 0 \bmod 4\,. \tag{4.16}$$

Therefore, imposing the equality of the lagrangians at $(y, 0)$ and $(0, y)$ relates the integers n_i that characterize the boundary conditions for the various fields in such a way that the theory is invariant under (4.9).

Finally, as in the scalar case, equality of the lagrangians at the (y, L) and (L, y) boundaries implies that the theory is invariant under the $Z_2^{\rm AP}$ transformation

$$\Psi^i_\pm(x^\mu, x^4, x^5) \mapsto (-1)^{l^\pm}\Psi^i_\pm(x^\mu, x^4, x^5)\,, \tag{4.17}$$

where the same $l^\pm$ applies to both four-dimensional chiralities L and R belonging to a given six-dimensional fermion $\Psi^i_\pm$.

In general, quantum loops contain divergences which correspond to localized counterterms at the points $(0, 0)$ and (L, L). This is similar to the situation of a fifth dimension compactified on the interval (the S^1/Z_2 orbifold), where localized operators are generated at the boundaries [24]. Furthermore, the six-dimensional theory could include such localized operators at tree level, with coefficients determined by the matching, at the six-dimensional cutoff scale Λ, between the compactified six-dimensional theory and its UV completion. One may worry that such operators could violate the symmetries discussed above. We note, however, that by virtue of their locality such operators are still tightly constrained by the six-dimensional spatial symmetries. What is special about the "fixed points" $(0,0)$ and (L,L) is that they correspond to the location of physical objects, or "branes", with attributes such as tension, etc. In particular, their presence allows to distinguish the four dimensions parallel to the brane from the two dimensions transverse to it, breaking the six-dimensional Lorentz invariance.[2] The key point, however, is that in our compactification the space in the vicinity of the "branes" has a rotational symmetry in the transverse dimensions. In fact, the brane locations correspond to conical singularities, with deficit angles of $3\pi/2$. Thus, the brane-localized operators should be explicitly $\mathrm{SO}(3,1)\times\mathrm{SO}(2)$ invariant. Since our argument above, eqs. (4.13)–(4.16), was based only on the SO(2) rotational symmetry, we conclude as before that its Z_8 subgroup is an exact symmetry of the compactified theory. The $Z_2^{\rm AP}$ symmetry associated with fields with $l^\pm = 1$ is clearly also a symmetry of the localized operators. The other corners of the square, $(L,0)$ and $(0,L)$, which are identified, have a conical singularity with a deficit angle of π, and the SO(2) rotational symmetry also ensures that any operators localized there are invariant under Z_8.

We have shown that the fact that there are eight possible boundary conditions that allow a nontrivial solution to the equation of motion leads to a $Z_8\times Z_2^{\rm AP}$ symmetry. Under the Z_8 symmetry a fermion of chirality $+_R$ or $-_L$ has charge $1/2+n$, a fermion of chirality $-_R$ or $+_L$ has charges $-1/2+n$, and a scalar has charge n, where $n=0,1,2,3$ defines the boundary conditions (2.35) on two adjacent sides of the square. Any operator is Z_8 invariant if the total charge is a multiple of four. Under the $Z_2^{\rm AP}$ symmetry, any field has a charge $l=0,1$, where l fixes the relative sign between the boundary conditions on the two pairs of identified sides of the square, as in eq. (2.35).

4.2 Kaluza-Klein parity

Next we will show that besides the $Z_8\times Z_2^{\rm AP}$ symmetry discussed so far there is another discrete symmetry that restricts the interactions among KK modes.

Let us first concentrate on theories where all fields have $l=0$ (but with no restriction on n). Under reflections (or, equivalently, rotations by π) about the center of the square $(L/2,L/2)$,

$$(x^4,x^5)\mapsto(L-x^4,L-x^5)\,, \tag{4.18}$$

the folding boundary conditions [see eqs. (2.35) and (3.26)] are invariant because the conditions that relate $l=0$ fields at $(0,y)$ and $(y,0)$ are interchanged with the ones at (L,y) and (y,L). Furthermore, the six-dimensional lagrangian is invariant under

$$\begin{aligned}\Phi(x^\mu,x^4,x^5)&\mapsto\Phi(x^\mu,L-x^4,L-x^5) &&(\text{scalars})\,,\\ \Psi_\pm(x^\mu,x^4,x^5)&\mapsto e^{-i\pi\Sigma_{45}/2}\,\Psi_\pm(x^\mu,L-x^4,L-x^5) &&(\text{fermions})\,,\end{aligned} \tag{4.19}$$

[2] The branes, and the operators localized on them, should be reparametrization invariant under both six-dimensional coordinate, as well as brane ("worldsheet") coordinate transformations. This is simply the statement that the brane, as a physical object, should have a geometric description. Therefore, there is a sense in which these operators are still constrained by local six-dimensional Lorentz transformations.

where the phases are given by $\mp i$ for $\Psi_{\pm R}$ and $\pm i$ for $\Psi_{\pm L}$. Thus, the six-dimensional action is also invariant under the above transformations. Consequently, the KK wavefunctions of type n are mapped into fields of the same type n. Explicitly, eq. (2.37) implies

$$f_n^{(j,k)}(L - x^4, L - x^5) = (-1)^{j+k+n} f_n^{(j,k)}(x^4, x^5) . \tag{4.20}$$

It follows that all bulk interactions, when decomposed into KK modes, give rise to interaction terms invariant under

$$\Phi^{(j,k)}(x^\mu) \mapsto (-1)^{j+k+n}\, \Phi^{(j,k)}(x^\mu) , \tag{4.21}$$
$$\Psi^{(j,k)}_{\pm R,L}(x^\mu) \mapsto (-1)^{j+k+q}\, \Psi^{(j,k)}_{\pm R,L}(x^\mu) , \tag{4.22}$$

where $\Phi^{(j,k)}(x^\mu)$, $\Psi^{(j,k)}_{\pm R,L}(x^\mu)$ are the KK modes as defined in eqs. (2.19) and (3.22), and the charges for the fermions, $q = \pm 1/2 + n$, are as given in eq. (4.9). Since the Z_8 symmetry discussed in the previous subsection requires that $\sum_i n_i$ and $\sum_i q_i$ are even for all interactions, the theory is actually invariant under the Z_2 transformation

$$\Upsilon^{(j,k)}(x^\mu) \mapsto (-1)^{j+k}\, \Upsilon^{(j,k)}(x^\mu) , \tag{4.23}$$

where Υ stands for either scalars or fermions. Due to the dependence on KK numbers of the above transformation, we refer to this symmetry as KK-parity, and we denote it by $Z_2^{\rm KK}$.

We should note that operators localized at the points $(0, 0)$ and (L, L) can potentially spoil this KK parity, unless they appear symmetrically on the two branes. The loop induced localized operators automatically satisfy this requirement, and therefore it is natural for the above theories to contain the KK-parity symmetry eq. (4.23), which implies that the lightest KK mode, with $(j, k) = (1, 0)$, is stable, and a good dark matter candidate [25, 26].

When fields with $l = 1$ are present, the folding boundary conditions (2.35) for $l = 1$ imply that under the reflection eq. (4.18), fields of type $(n, l = 1)$ are mapped into fields of type $(n + 2, l = 1)$. Explicitly,

$$f^{(j,k)}_{n,l=1}(L - x^4, L - x^5) = (-1)^{j+k+n} f^{(j,k)}_{n+2,l=1}(x^4, x^5) . \tag{4.24}$$

Therefore, in order for the transformation (4.18) to be a symmetry of a theory containing $l = 1$ fields, it is necessary that both $(n, l = 1)$ and $(n + 2, l = 1)$ fields be present. We note, however, that independently of whether the transformation (4.18) is a symmetry of the theory, when fields with $l = 1$ are present the lightest KK mode is a $(j, k) = (1/2, 1/2)$ state, which due to the $Z_2^{\rm AP}$ symmetry discussed in the previous subsection [see eqs. (4.5) and (4.17)] is stable and can only be pair produced.

5. Interactions of the Kaluza-Klein modes

It is instructive to analyze how the symmetries discussed in the previous section are realized in the KK picture. To this end, one has to integrate products of KK wave-functions over the square. Recall that the KK wave-functions for both scalars and fermions, $f_n^{(j,k)}(x^4, x^5)$, are given by eq. (2.37), with $j = 0$ only if $n = k = 0$, and $j > 0$ otherwise. These have the property

$$(\partial_4 \pm i\partial_5)\, f_n^{(j,k)}(x^4, x^5) = \frac{i\pi}{L}(j \pm ik) f_{n\mp 1}^{(j,k)}(x^4, x^5)\,, \tag{5.1}$$

and therefore, whether or not an operator involves derivatives, its integral over the square is of the type

$$\frac{1}{L^2}\int_0^L dx^4 \int_0^L dx^5\, f_{n_1}^{(j_1,k_1)} \cdots f_{n_r}^{(j_r,k_r)} = \frac{2^{2-r}\Delta_{n_1\cdots n_r}^{(j_1,k_1)\cdots(j_r,k_r)}}{(1+\delta_{j_1,0})\cdots(1+\delta_{j_r,0})} \tag{5.2}$$

where r is the total number of fields that appear in the operator. The Z_8 symmetry restricts the products of KK wave-functions to satisfy

$$n_1 + \cdots + n_r = 0 \bmod 4\,. \tag{5.3}$$

In what follows we restrict attention only to fields with folding boundary conditions of the type $l = 0$, for which j and k are integers. In order to compute $\Delta_{n_1\cdots n_r}^{(j_1,k_1)\cdots(j_r,k_r)}$ it is convenient to observe that the basic integral is

$$I(j,k) \equiv \frac{1}{L^2}\int_0^L dx^4 \int_0^L dx^5\, e^{i\pi(jx^4+kx^5)/L}\,, \tag{5.4}$$

which satisfies

$$I(j,k) + I(-j,-k) + I(k,-j) + I(-k,j) = 4\delta_{j,0}\delta_{k,0}\,. \tag{5.5}$$

This property follows from the analytical continuation of the $I(j,k)$ integrals to the $-L < x^4, x^5 < L$ region.

For a trilinear interaction ($r = 3$) the result is

$$\begin{aligned}
\Delta_{n_1,n_2,n_3}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)} = {} & 7\delta_{j_1,0}\delta_{j_2,0}\delta_{j_3,0} + \delta_{j_1+j_2,j_3}\delta_{k_1+k_2,k_3}\, e^{in_3\pi} + \\
& +\delta_{j_2+j_3,j_1}\delta_{k_2+k_3,k_1}\, e^{in_1\pi} + \delta_{j_3+j_1,j_2}\delta_{k_3+k_1,k_2}\, e^{in_2\pi} + \\
& +\delta_{j_1+k_2,j_3}\delta_{j_2+k_3,k_1}\, e^{i(n_2/2+n_3)\pi} + \delta_{j_1+k_3,j_2}\delta_{j_3+k_2,k_1}\, e^{i(n_3/2+n_2)\pi} + \\
& +\delta_{j_2+k_3,j_1}\delta_{j_3+k_1,k_2}\, e^{i(n_3/2+n_1)\pi} + \delta_{j_2+k_1,j_3}\delta_{j_1+k_3,k_2}\, e^{i(n_1/2+n_3)\pi} + \\
& +\delta_{j_3+k_2,j_1}\delta_{j_2+k_1,k_3}\, e^{i(n_2/2+n_1)\pi} + \delta_{j_3+k_1,j_2}\delta_{j_1+k_2,k_3}\, e^{i(n_1/2+n_2)\pi}\,.
\end{aligned} \tag{5.6}$$

Note that the first term is nonzero only when all the KK modes are (0,0), while the other nine terms are generic, describing interactions of various KK modes. In general, for an interaction involving r fields, there are $t_r = (2^{r-1}-1)^2$ generic terms and a term applying only to zero-modes of coefficient $4^{r-1} - t_r$.

A particular case of interest for phenomenological applications is the interaction of a number of zero modes with two higher modes. The integral over the square gives simply

$$\frac{1}{L^2}\int_0^L dx^4 \int_0^L dx^5\, f_n^{(j_1,k_1)} f_{-n}^{(j_2,k_2)} f_0^{(0,0)} \cdots f_0^{(0,0)} = \delta_{j_1,j_2}\delta_{k_1,k_2} e^{-in\pi}\,. \tag{5.7}$$

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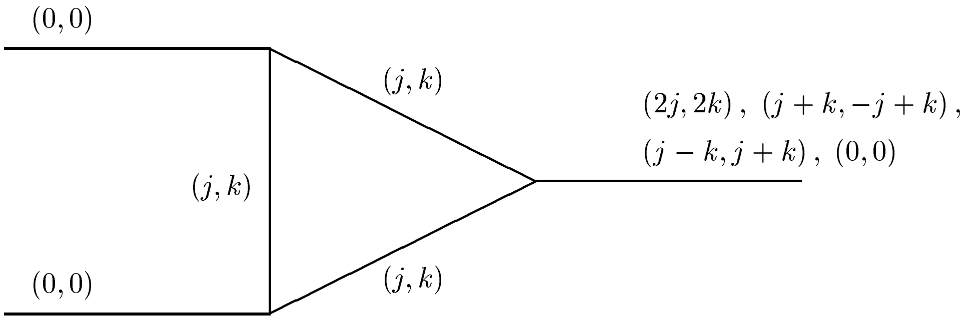

Figure 2: Effective interaction of two zero modes with a higher KK mode.

Eq. (5.6) provides the tree-level selection rules for a trilinear interaction. Given the KK numbers of two of the fields, (j_1, k_1) and (j_2, k_2), the KK numbers of the third field are given by one of the following pairs:

$$\begin{aligned}(j_3, k_3) = &(j_1 + j_2\,,\, k_1 + k_2)\,,\ (j_1 - j_2\,,\, k_1 - k_2)\,,\ (-j_1 + j_2\,,\, -k_1 + k_2)\,,\\ &(j_1 + k_2\,,\, k_1 - j_2)\,,\ (j_1 - k_2\,,\, k_1 + j_2)\,,\ (k_1 + j_2\,,\, -j_1 + k_2)\,,\\ &(-k_1 + j_2\,,\, j_1 + k_2)\,,\ (k_1 - k_2\,,\, -j_1 + j_2)\,,\ (-k_1 + k_2\,,\, -j_1 + j_2)\,. \end{aligned} \tag{5.8}$$

All these choices are consistent with the KK parity discussed in the previous section, as they satisfy the condition that both $j_1 + j_2 + j_3$ and $k_1 + k_2 + k_3$ are even. However, not all values for (j_3, k_3) allowed by KK parity are generated by the bulk interactions at tree level. As in the five-dimensional case discussed in ref. [22], the other values for (j_3, k_3) are generated by loops. For example, there is no tree-level coupling of two zero modes to a higher mode, while at one loop there is a coupling of two zero modes to the $(2j, 2k)$, $(j + k, -j + k)$ and $(j - k, j + k)$ modes of $n = 0$ fields, for any integer values of j, k, as shown in figure 2.

6. Connection to orbifold theories

We now mention the relation between the compactification on the folded square that we are studying and orbifold compactifications. As has become clear from the discussion in subsection 4.1 there is a close connection between the theory on the folded square and rotations by $\pi/2$ in an extended theory valid on the larger square $-L \le x^4, x^5 \le L$ (or even the full plane). In fact, the theories we are studying can be obtained by starting from fields defined on the whole six-dimensional space subject to periodic or anti-periodic boundary conditions

$$\Phi^{[p]}(x^\mu, x^4 + 2L, x^5) = \Phi^{[p]}(x^\mu, x^4, x^5 + 2L) = \Phi^{[p]}(x^\mu, x^4, x^5)\,, \tag{6.1}$$

or

$$\Phi^{[a]}(x^\mu, x^4 + 2L, x^5) = \Phi^{[a]}(x^\mu, x^4, x^5 + 2L) = -\Phi^{[a]}(x^\mu, x^4, x^5)\,. \tag{6.2}$$

Fields that satisfy the periodic boundary condition are identified with our $l = 0$ fields, while those that satisfy the anti-periodic one correspond to our $l = 1$ fields. In order to consistently impose the anti-periodic boundary condition, eq. (6.2), it is necessary that the underlying lagrangian has a parity symmetry $\Phi^{[a]} \mapsto -\Phi^{[a]}$, which corresponds to the $Z_2^{\rm AP}$ parity found in subsection 4.1 [see eqs. (4.5) and (4.17)].

Next, one can use the fact that the theory so defined has a symmetry under rotations by $\pi/2$ about $(x^4, x^5) = (0, 0)$ to perform a further "orbifold" projection. For scalars it reads

$$\Phi(x^\mu, -x^5, x^4) = e^{i\theta}\Phi(x^\mu, x^4, x^5)\,, \tag{6.3}$$

where an internal U(1) symmetry has been assumed that allows us to identify fields at rotated points up to a phase. For fermions, the orbifold projection is

$$e^{-i(\pi/2)\Sigma_{45}/2}\Psi(x^\mu, -x^5, x^4) = e^{i\theta}\Psi(x^\mu, x^4, x^5)\,. \tag{6.4}$$

We have explicitly shown in our derivation in sections 2 and 3 that from all possible phases that are allowed a priori, only those satisfying $\theta = n\pi/2$ with $n = 0, 1, 2, 3$ allow for nontrivial solutions. The above comments further show that the compactification on the folded square is equivalent to a T^2/Z_4 field-theory orbifold, where the Z_4 corresponds to rotations by $\pi/2$ in the plane of the compactified dimensions and the fields on the torus T^2 may be periodic ($l = 0$) or anti-periodic ($l = 1$). Consequently, the KK wave-functions for scalars[3] and fermions shown in eqs. (2.37) and (3.25), as well as the symmetries discussed in section 4 are identical with the ones that can be derived by starting from the T^2/Z_4 compactification.

7. Summary

We conclude with a recapitulation of our main results regarding six-dimensional field theories with two dimensions compactified on a square. The boundary conditions prescribe the identification of two pairs of adjacent sides of the square, and the equality of lagrangian densities at identified points.

A field has values at pairs of identified points which may differ by a symmetry transformation. In the case of a complex scalar there is a U(1) symmetry, so that the field values may differ by a phase. In general, the phase difference of a pair of adjacent sides need not be equal to the phase difference of the other pair. We refer to these as the "folding" boundary conditions, and to the compactification in general as the "folded square". We have shown that the compactification is smooth everywhere, in the sense that the derivative of the field in the direction perpendicular to the identified boundaries is continuous up to a phase.

The field equation has nontrivial solutions only if the phase that relates field values at identified points is $n\pi/2$, $n = 0, 1, 2, 3$. In addition, the phases associated with the two pairs of identified sides may differ by $l\pi$, $l = 0, 1$. The most general folding boundary conditions that allow solutions to the field equation for a complex scalar, labeled by the two integers n and l, is given by eq. (2.35).

[3]The KK wave-functions derived in section 2 [see eq. (2.37)] agree with the ones given in eq. (3.12) of ref. [4] except for a typo in that equation ($t^3 f_{p_y,-p_z}$ should read $t^3 f_{p_z,-p_y}$, and $t^3 f_{p,0}$ should read $t^3 f_{0,p}$).

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It turns out that a folded square of size L is equivalent to the compactification on a T^2/Z_4 orbifold where the T^2 is a torus of size $2L$. For a theory involving arbitrary interactions, the Z_4 symmetry is a requirement for the theory on the orbifold, whereas on the folded square the same symmetry arises from the equality of the lagrangian densities at identified points, combined with local six-dimensional Lorentz invariance. The four values of n are the possible values of the charge of the scalars under the Z_4 symmetry, while the two values of l correspond to periodic ($l = 0$) and anti-periodic ($l = 1$) boundary conditions on T^2. The equality of the lagrangian densities at identified points also implies the existence of a Z_2 symmetry, which we label $Z_2^{\rm AP}$, under which the fields have charge l.

The wave functions of the KK modes, $f_n^{(j,k)}(x^4, x^5)$, can be written as the sum of two cosines, as in eq. (2.37). They depend on two KK numbers, $j, k \geq 0$, which are integers for $l = 0$ and half-integers for $l = 1$. Only the fields with $n = l = 0$ have a zero mode, i.e., $j = k = 0$. The completeness condition allows all the $k = 0$ states but none of the $j = 0$ ones, except for the zero-mode.

Most of the above conclusions apply to fermions as well, with additional intricacies related to chirality. In the case of a six-dimensional chiral fermion, the Dirac equation has non-trivial solutions only if the folding boundary conditions for the left- and right-handed four-dimensional chiralities are different. This is a key property that allows the embedding of a four-dimensional chiral theory, such as the standard model, into a six-dimensional theory with bulk fermions. Specifically, the left- and right-handed components of a fermion of six-dimensional chirality $\pm$ have folding boundary conditions with values for n that differ by ± 1 mod 4. In particular, if the $+_L$ or $-_R$ ($-_L$ or $+_R$) chirality has a zero mode, then the $+_R$ or $-_L$ ($+_L$ or $-_R$) chirality has a wave function of type $n = 3$ ($n = 1$). The phases of the KK wave functions for the left- and right-handed components are also correlated, their difference being given by the complex phase of $k \pm ij$ in the case of six-dimensional chirality $\pm$.

One difference compared to scalar theories is that in the presence of fermions the Z_4 symmetry of the action is promoted to a Z_8 symmetry. This is a consequence of the six-dimensional Lorentz symmetry, or more precisely of the invariance under rotations by $\pi/2$ in the (x^4, x^5) plane. In the context of the T^2/Z_4 orbifold [1], the Z_8 symmetry is the group of rotations by $\pi/2$ around the center of T^2. From the point of view of the folded square, Z_8 is an internal symmetry, with fermions carrying discrete charge $q = \Sigma_{45}/2 + n$, where Σ_{45} has eigenvalue ∓ 1 for the $\pm_L$ chiralities, and ± 1 for the $\pm_R$ chiralities. For a scalar with folding boundary conditions of type n, the Z_8 charge is n. The Z_8 symmetry requires that all operators in the four-dimensional effective theory have a Z_8 charge given by 0 mod 4. In ref. [1], it has been shown that for the standard model in six dimensions the Z_8 symmetry ensures a lifetime for the proton longer than the current experimental bounds, even in the presence of baryon number violation at the TeV scale. Another implication is that Majorana masses are forbidden (the implications for neutrino masses are discussed in ref. [17]). Although we restricted attention in this paper only to fermions and scalars, the derivation of the Z_8 symmetry is based on general arguments regarding the six-dimensional Lorentz symmetry, which hold in the presence of fields of any spin. The interesting case of gauge fields will be analyzed in ref. [27].

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In theories where all fields satisfy boundary conditions with $l = 0$, the above $Z_2^{\rm AP}$ symmetry becomes trivial. In this situation, however, the folded square compactification has one more symmetry, namely invariance under reflection with respect to the center of the square. In contrast to the above $Z_8 \times Z_2^{\rm AP}$ symmetry, which assigns a unique charge to the whole tower of KK modes belonging to a six-dimensional field (of given four-dimensional chirality, in the case of fermions), the symmetry under reflection distinguishes between KK modes. Therefore, this symmetry is a KK parity, and we label it by $Z_2^{\rm KK}$. A (j,k) mode changes sign (remains invariant) under reflection if $j+k$ is odd (even). The $Z_2^{\rm KK}$ symmetry is similar to the KK parity of five-dimensional theories compactified on the S^1/Z_2 orbifold, which is invariance under reflection with respect to the center of the interval. In six dimensions, however, reflection is part of the Lorentz symmetry, namely it is a rotation by π. Therefore, any field theory on the folded square has an exact $Z_8 \times Z_2$ symmetry, as a consequence of the equality of lagrangian densities at identified points and of the six-dimensional Lorentz symmetry. If fields that satisfy boundary conditions with $l = 1$ are present, then $Z_2 = Z_2^{\rm AP}$. When all fields satisfy boundary conditions with $l = 0$, then $Z_2 = Z_2^{\rm KK}$. In either case, the Z_2 symmetry guarantees the stability of the lightest Kaluza-Klein mode, which may play the role of dark matter [25, 26].

The KK parity has important consequences on theories defined on the folded square, since it provides a selection rule for any interaction: the sum of all KK numbers entering a vertex should be even. The bulk interactions generate at tree level only a subset of these vertices [see eq. (5.8) for the case of a trilinear interaction], but quantum loops generate the other ones, similarly to the five-dimensional case studied in [22].

In the compactification on the folded square, the corners of the square correspond to conical singularities. Two of the points are identified, while the other two points are the fixed points of the T^2/Z_4 orbifold. Typically, loops generate operators localized at these points with divergent coefficients. This suggests that physics at the cut-off scale could generate additional contributions to the operators localized at these points. Nevertheless, such contributions are expected to be Z_8 invariant because the space in the vicinity of the corners has rotational symmetry. Moreover, as long as the underlying dynamics that induces contributions at the cut-off scale does not distinguish between the fixed points, the $Z_2^{\rm KK}$ symmetry is also preserved by any localized operators. When fields satisfying boundary conditions with $l = 1$ are present, the $Z_2^{\rm KK}$ parity is generically broken by their boundary conditions, but a new $Z_2 = Z_2^{\rm AP}$ symmetry emerges, as mentioned above.

On the other hand, operators localized at the fixed points may perturb the KK spectrum. At tree level, the squared KK masses are given by $(j^2 + k^2)(\pi/L)^2$ plus the squared mass of the six-dimensional field. Further contributions induced by loops include both finite pieces due to bulk kinetic terms and divergent pieces due to kinetic terms localized at the fixed points, as shown in [22]. Likewise, contributions from physics above the cut-off scale to kinetic terms localized at the fixed points would modify the KK masses.

The folded square is a simple compactification of two extra dimensions that allows the existence of chiral fermions in the four-dimensional effective theory. Topologically, the boundary reduces to only a couple of points, and therefore this compactification has good

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prospects for being the low energy behavior of some underlying dynamics. Furthermore, the folded square has an intriguing symmetry structure. It would therefore be interesting to compactify various six-dimensional extensions of the standard model on the folded square.

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